AN ASYMPTOTIC ANALYSIS OF SPHERICALLY SYMMETRIC PERFECT FLUID SIMILARITY SOLUTIONS

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Abstract

The asymptotic properties of self-similar spherically symmetric perfect fluid solutions with equation of state $p = \alpha\mu$ $(-1 < \alpha < 1)$ are described. We prove that for large and small values of the similarity variable, $z = r/t$, all such solutions must have an asymptotic power-law form. They are associated either with an *exact* power-law solution, in which case the $\alpha > 0$ ones are asymptotically Friedmann, asymptotically Kantowski-Sachs or asymptotically static, or with an *approximate* power-law solution, in which case they are asymptotically quasi-static for $\alpha > 0$ or asymptotically Minkowski for $\alpha > 1/5$. We also show that there are solutions whose asymptotic behaviour is associated with *finite* values of z and which depend upon powers of lnz. These correspond either to a second family of asymptotic Minkowski solutions for $\alpha > 1/5$ or to solutions that are asymptotic to a central singularity for $\alpha > 0$. The asymptotic form of the solutions is given in all cases, together with the number of associated parameters. This forms the basis for a complete classification of all $\alpha > 0$ self-similar solutions. There are some other asymptotic power-law solutions associated with negative α, but the physical significance of these is unclear.

1   Introduction

Spherically symmetric solutions to Einstein's equations admitting a self-similarity of the first kind are characterized by the fact that the spacetime possesses a homothetic Killing vector. This means that they can be put into a form in which

every dimensionless variable is a function of some dimensionless combination of the cosmic time coordinate t and the comoving radial coordinate r. Such solutions have attracted considerable attention in recent decades. This is partly for mathematical reasons, since the governing equations reduce to ordinary differential equations from partial differential equations. In particular, in the case of a perfect fluid, the only barotropic equation of state compatible with the similarity assumption is one of the form $p = \alpha\mu$, where α is a constant, and the equations then simplify still further. However, it is also for physical reasons, since such solutions play a crucial role in many cosmological and astrophysical contexts and are important in the context of the "critical" phenomena discovered in recent gravitational collapse calculations (Evans and Coleman 1994, Carr et al. 1999, Neilsen and Choptuik 1998). Indeed the "similarity hypothesis" proposes that spherically symmetric solutions may naturally evolve to similarity form in a wide variety of situations. The arguments for this have been reviewed by Carr & Coley (1999a).

The possibility that self-similar models may be singled out in this way from more general spherically symmetric solutions means that it is essential to study the full family of such solutions. Two main approaches have been used in such studies. The first uses the "comoving" approach, pioneered by Cahill & Taub (1971), in which the coordinates are adapted to the fluid 4-velocity vector and the solutions are parametrized by the similarity variable $z = r/t$. At a given value of t, this specifies the spatial profile of the various quantities. At a given value of r (i.e. for a given fluid element), it specifies their time evolution. The second uses the "homothetic" approach, introduced by Bogoyavlensky (1977). In this, the coordinates are adapted to the homothetic vector and solutions can be treated as orbits in a compactified 3-dimensional phase space.

The purpose of this paper is to delineate all possible asymptotic behaviours of spherically symmetric $p = \alpha\mu$ similarity solutions using the comoving approach. This provides the first step in our "complete" classification of all such solutions with $\alpha > 0$ in another paper (Carr & Coley 1999b; CC). The classification is complete, subject to the requirement that the solutions are physical everywhere and do not contain shocks. However, our claim for completeness in that paper is based on the assumption that all solutions must have an asymptotic form in which any dimensionless quantity has a power-law dependence on z at large and small values of z or a power-law dependence on lnz at finite z (whenever this corresponds to zero or infinite distance from the origin). We do not *prove* this assumption in CC but take it as our starting point. We then find that the behaviour of solutions in the asymptotic limits can only take one of a few simple forms for $\alpha > 0$: asymptotically Friedmann, asymptotically Kantowski-Sachs,

asymptotically Minkowski (for $\alpha > 1/5$), asymptotically singular or what we term asymptotically "quasi-static".

In this paper we also obtain the asymptotic behaviours but in a different way and, in the process, we provide a rigorous *proof* of the "power-law" assumption in the physically important positive α case. This paper therefore crucially complements the classification paper. Indeed we sometimes refer to equations in CC below. Both papers end up with the same asymptotic behaviours for $\alpha > 0$ but the analysis in this paper is more rigorous and more comprehensive. In addition, this paper also covers the negative α case. However, we do not consider the physical significance of any of the solutions in this paper. The positive α solutions are discussed in more detail elsewhere (CC, Carr et al. 1999). The physical significance of the negative α solutions are less clear; some applications are discussed in Carr & Coley (1999a) and we note that for $-1 < \alpha < -1/3$ the asymptotically Friedmann (AF) solutions (for example – see Table 1) are inflationary (Olive, 1990)

It should be stressed that Goliath et al. (1998a & b; GNU) have also implicitly provided a classification of spherically symmetric self-similar solutions with positive α using the homothetic approach. However, the nature of their classification is rather different in that they emphasize the equilibrium points of the associated dynamical system. These always lie on the *boundary* of the 3-dimensional phase space, whereas the condition $z \to \infty$ emphasized by CC usually corresponds to the *interior* of the phase space and so does not play a crucial role in the GNU analysis (for example, there are no equilibrium points corresponding to CC's asymptotically quasi-static solutions which occur as $|z| \to \infty$). However, some of these equilibrium points do have a straightforward connection to CC's analysis. Thus what GNU term the "F point" corresponds to solutions which are asymptotically Friedmann as $z \to \infty$, their "C^0 point" corresponds to solutions which are asymptotically Friedmann as $z \to 0$ (or regular at the origin), their "T point" corresponds to the static solution, and their "K point" corresponds to (asymptotically Kasner) solutions with a central singularity (occuring at a finite value of z). The two approaches are therefore complementary and the value of using a combined approach has been emphasized by Carr et al. (1999).

The outline of this paper is the following. In Section 2 we introduce the general spherically symmetric similarity equations for a perfect fluid and highlight the features necessary for our analysis. In Section 3 we provide the analysis itself; this comprises an exhaustive case-by-case consideration of the different possible behaviours of various key functions. In Section 4 we draw some general conclusions and point out some more subtle issues which are not covered explictly in Section 3.

2 Spherically Symmetric Similarity Solutions

In the spherically symmetric situation one can introduce a time coordinate t such that surfaces of constant t are orthogonal to fluid flow lines and comoving coordinates (r, θ, ϕ) which are constant along each flow line. The metric can then be written in the form

$$ds^2 = e^{2\nu}\, dt^2 - e^{2\lambda}\, dr^2 - R^2\, d\Omega^2, \quad d\Omega^2 \equiv d\theta^2 + \sin^2\theta\, d\phi^2 \qquad (2.1)$$

where ν, λ and R are functions of r and t. For a perfect fluid the Einstein equations are

$$G^{\mu\nu} = 8\pi[(\mu + p)U^\mu U^\nu - p\, g^{\mu\nu}] \qquad (2.2)$$

where $\mu(r, t)$ is the energy density, $p(r, t)$ the pressure, $U^\mu = (e^{-\nu}, 0, 0, 0)$ is the comoving fluid 4-velocity, and we choose units in which $c = G = 1$. The equations have a first integral

$$m(r, t) = \tfrac{1}{2}R \left[1 + e^{-2\nu}\left(\frac{\partial R}{\partial t}\right)^2 - e^{-2\lambda}\left(\frac{\partial R}{\partial r}\right)^2 \right] \qquad (2.3)$$

and this can be interpreted as the mass within comoving radius r at time t:

$$m(r, t) = 4\pi \int_0^r \mu R^2 \frac{\partial R}{\partial r'}\, dr'. \qquad (2.4)$$

Unless $p = 0$, this quantity decreases with increasing t because of the work done by the pressure.

A self-similar solution is one in which the spacetime admits a homothetic Killing vector $\boldsymbol{\xi}$ such that

$$\xi_{\mu;\nu} + \xi_{\nu;\mu} = 2g_{\mu\nu}. \qquad (2.5)$$

This means that the solution is unchanged by a transformation of the form $t \to at$, $r \to ar$ for any constant a. Solutions of this type were first investigated by Cahill & Taub (1971), who showed that by a suitable coordinate transformation they can be put into a form in which all dimensionless quantities such as ν, λ,

$$S \equiv \frac{R}{r}, \quad M \equiv \frac{m}{R}, \quad P \equiv pR^2, \quad W \equiv \mu R^2 \qquad (2.6)$$

are functions only of the dimensionless variable $z \equiv r/t$. Then we have

$$\frac{\partial}{\partial t} = -\frac{z^2}{r}\frac{d}{dz}, \quad \frac{\partial}{\partial r} = \frac{z}{r}\frac{d}{dz}, \qquad (2.7)$$

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so the field equations reduce to a set of ordinary differential equations in z. Another important quantity is the function

$$V(z) = e^{\lambda - \nu} z, \tag{2.8}$$

which represents the velocity of the surfaces of constant z relative to the fluid. These surfaces have the equation $r = zt$ and therefore represent a family of spheres moving through the fluid. The spheres contract relative to the fluid for $z < 0$ and expand for $z > 0$. Special significance is attached to values of z for which $|V| = 1$ and $M = 1/2$. The first corresponds to a Cauchy horizon (either a black hole event horizon or a cosmological particle horizon), the second to a black hole or cosmological apparent horizon.

The only barotropic equation of state compatible with the similarity ansatz is one of the form $p = \alpha\mu$ ($-1 \leq \alpha \leq 1$). It is convenient to introduce a dimensionless function $x(z)$ defined by

$$x(z) \equiv (4\pi\mu r^2)^{-\alpha/(1+\alpha)}. \tag{2.9}$$

The conservation equations $T^{\mu\nu}{}_{;\nu} = 0$ can then be integrated to give

$$e^{\nu} = \beta x z^{2\alpha/(1+\alpha)} \tag{2.10}$$

$$e^{-\lambda} = \gamma x^{-1/\alpha} S^2 \tag{2.11}$$

where β and γ are integration constants. The remaining field equations reduce to a set of ordinary differential equations in x and S:

$$\ddot{S} + \dot{S} + \left(\frac{2}{1+\alpha}\frac{\dot{S}}{S} - \frac{1}{\alpha}\frac{\dot{x}}{x}\right)[S + (1+\alpha)\dot{S}] = 0, \tag{2.12}$$

$$\left(\frac{2\alpha\gamma^2}{1+\alpha}\right)S^4 + \frac{2}{\beta^2}\frac{\dot{S}}{S}x^{(2-2\alpha)/\alpha}z^{(2-2\alpha)/(1+\alpha)} - \gamma^2 S^4\frac{\dot{x}}{x}\left(\frac{V^2}{\alpha} - 1\right) = (1+\alpha)x^{(1-\alpha)/\alpha}, \tag{2.13}$$

$$M = S^2 x^{-(1+\alpha)/\alpha}\left[1 + (1+\alpha)\frac{\dot{S}}{S}\right], \tag{2.14}$$

$$M = \frac{1}{2} + \frac{1}{2\beta^2}x^{-2}z^{2(1-\alpha)/(1+\alpha)}\dot{S}^2 - \frac{1}{2}\gamma^2 x^{-(2/\alpha)}S^6\left(1 + \frac{\dot{S}}{S}\right)^2, \tag{2.15}$$

where the velocity function is given by

$$V = (\beta\gamma)^{-1}x^{(1-\alpha)/\alpha}S^{-2}z^{(1-\alpha)/(1+\alpha)} \tag{2.16}$$

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and an overdot denotes zd/dz.

We can envisage how these equations generate solutions by working in the 3-dimensional $(x, S, \dot{S})$ space (Carr and Yahil 1990). At any point in this space, for a fixed value of α, eqns (2.14) and (2.15) give the value of z; eqn (2.13) then gives the value of $\dot{x}$ unless $|V| = \sqrt{\alpha}$ and eqn (2.12) gives the value of $\ddot{S}$. Thus the equations generate a vector field $(\dot{x}, \dot{S}, \ddot{S})$ and this specifies an integral curve at each point of the 3-dimensional space. Each curve is parametrized by z and represents one particular similarity solution. This shows that, for a given equation of state parameter α, there is a 2-parameter family of spherically symmetric similarity solutions. Special significance is attached to points where $|V| = \sqrt{\alpha}$, which specifies a 2-dimensional surface in $(x, S, \dot{S})$ space. This corresponds to a sonic point, so there can be a discontinuity in the pressure gradient (i.e. the value of $\dot{x}$) here. The requirement that solutions be regular at the sonic point (in the sense that they can be extended beyond there) places severe restrictions on the nature of the solutions (Bogoyavlensky 1977, Bicknell & Henriksen 1978, Carr & Yahil 1990, Ori & Piran 1990) but these restrictions are not relevant to the present asymptotic analysis.

3   Asymptotic Classification

In this section we will provide a complete analysis of spherically symmetric similarity solutions in the limits $R \to 0$ and $R \to \infty$. (We will assume $z > 0$ throughout since the $z < 0$ solutions are just the time reverse of the $z > 0$ ones.) This may correspond to $z \to 0$, $z \to \infty$ or $z \to z_*$ (finite) since the correspondence between the comoving radial coordinate r and the physical distance R may not be straightforward. In particular, $r \to 0$ need not imply $R \to 0$ (eg. some dust solutions), $R \to 0$ need not imply $r \to 0$ (eg. black hole singularities), $r \to \infty$ need not imply $R \to \infty$ (eg. the Kantowski-Sachs solution) and $R \to \infty$ need not imply $r \to \infty$ (eg. asymptotically Minkowski solutions). We categorize solutions according to whether $V \to 0$, $V \to \infty$ or $V \to V_*$ (finite) in each of the z limits. A key role is played by eqn (2.13), which can be written as

$$\frac{\dot{x}}{x} = \frac{2V^2\alpha}{V^2 - \alpha}\frac{\dot{S}}{S} + \frac{2\alpha^2}{1+\alpha}\frac{1}{V^2 - \alpha} - \xi, \quad \xi \equiv \frac{\alpha(1+\alpha)}{\gamma^2}\frac{S^{-4}x^{(1-\alpha)/\alpha}}{V^2 - \alpha}. \qquad (3.1)$$

One has different possible behaviours according to whether $\xi \to 0$, $\xi \to \infty$ or $\xi \to \xi_*$ (finite). In principle we need to analyse 27 cases, corresponding to the three limiting behaviours of each of z, V and ξ. However, in practice, many of these do not lead to a consistent solution. In the discussion below, we usually only derive the self-consistent ones.

Our procedure is as follows: for each limiting value of z, V and ξ, we use eqn (3.1) to eliminate $\dot{x}/x$ in eqn (2.12) and thereby obtain a second order ODE for S. Not every solution of this ODE may be consistent with the other equations, so we first confirm that ξ has the correct limiting value and then check that the integral condition associated with eqns (2.14, 2.15) is satisfied. This procedure not only identifies the possible asymptotic behaviours but also indicates the number of free parameters in each case. The results are summarized in Table 1.

	$z \to 0$	$z \to z_*$	$z \to \infty$
$V \to 0$	AKS (0/1) $(1 > \alpha > -1/3)$ AF (1) $(1 > \alpha > -1/3)$ ES (0) $(1 > \alpha > 0)$ AX (1) $(-0.17 < \alpha < -1/7)$	No soln	AF (1) $(-1 < \alpha < -1/3)$ AKS (1) $(-1 < \alpha < -1/3$ AX (1) $(1 > \alpha > -1/7)$
$V \to V_*$	No soln	AM (2) $(1 > \alpha > 1/5)$ AKS (1) $(1 > \alpha > 0)$	AM (1) $(1 > \alpha > 1/5)$ AM (1) $(-1 < \alpha < 0)$
$V \to \infty$	AKS (1) $(-1 < \alpha < -1/3)$ AF (1) $(-1 < \alpha < -1/3)$	AK (2) $(1 > \alpha > -1/3)$	AQS (2) $(1 > \alpha > 0)$ AKS (0/1) $(1 > \alpha > -1/3$ AF (1) $(1 > \alpha > -1/3)$ AY (1) $(-1 < \alpha < 0)$

Table 1. *This summarizes the asymptotic (A) behaviour of spherically symmetric perfect fluid similarity solutions for different ranges of α. The number of arbitrary parameters for each solution is given in parentheses. KS, F, M, K, ES and QS denote Kantowski-Sachs, Friedmann, Minkowski, Kasner, Exact Static and Quasi-Static respectively. X and Y correspond to the two new solutions in which either V or M is negative. The AKS solution is described by one parameter for $\alpha > 0$ and $-1 < \alpha < -1/3$ but there is only the exact KS solution for $-1/3 < \alpha < 0$; V and M are positive only for $-1 < \alpha < -1/3$.*

In the following analysis a differential equation of the form

$$\ddot{S} + a\dot{S} + b\frac{\dot{S}^2}{S} = 0, \tag{3.2}$$

where a and b are constants, will frequently arise. When $a \neq 0$, we can divide through by S and integrate this equation to obtain

$$\dot{S}S^b \sim z^{-a}. \tag{3.3}$$

If $b \neq -1$, we can then integrate a second time to obtain the exact solution

$$S = (c_1 + c_2 z^{-a})^{1/(1+b)}, \tag{3.4}$$

where c_1 and c_2 are integration constants. The dominant asymptotic solution is then determined by the sign of a. Note that the first order perturbation to the dominant solution (which is not derived in the present analysis) will often be larger than the subdominant solution.

- $V \to 0$ as $z \to 0$

In this case, eqn (3.1) becomes

$$\frac{\dot{x}}{x} = -\frac{2\alpha}{1+\alpha} - \xi, \quad \xi \equiv -\frac{1+\alpha}{\gamma^2} S^{-4} x^{(1-\alpha)/\alpha}. \tag{3.5}$$

If $\xi \to 0$, this implies $x \sim z^{-2\alpha/(1+\alpha)}$, so eqn (2.12) gives

$$\ddot{S} + \left(\frac{5+3\alpha}{1+\alpha}\right)\dot{S} + \frac{2\dot{S}^2}{S} + \frac{2S}{1+\alpha} = 0. \tag{3.6}$$

Defining $\sigma(z) \equiv zS(z)$, this becomes

$$\ddot{\sigma} - \left(\frac{1+3\alpha}{1+\alpha}\right)\dot{\sigma} + \frac{2\dot{\sigma}^2}{\sigma} = 0. \tag{3.7}$$

This differential equation is of the form of (3.2) and, from eqn (3.4), we therefore obtain the exact solution

$$S = z^{-1}[c_1 + c_2 z^{(1+3\alpha)/(1+\alpha)}]^{1/3}. \tag{3.8}$$

For $c_1 = 0$, we obtain the exact power-law solution

$$S = Az^{-2/3(1+\alpha)}, \quad x = Bz^{-2\alpha/(1+\alpha)}, \tag{3.9}$$

where the constants A and B are constrained by eqns (2.14, 2.15) and can be taken to be 1 with a suitable choice of the scaling constants β and γ in eqns (2.10, 2.11) [see CC eqn (3.10)]. This corresponds to the k=0 Friedmann model. Note that $V \to 0$ and $\xi \to 0$ (as assumed) providing $\alpha > -1/3$. As indicated in Table (1) and discussed in CC, there is also a 1-parameter family of solutions asymptotic to this. This involves modifying the constants A and B, as well as adding higher order terms in z. However, demonstrating this depends on a higher order analysis than presented here.

For $c_2 = 0$, one obtains another power-law solution

$$S = Az^{-1}, \quad x = Bz^{-2\alpha/(1+\alpha)}, \tag{3.10}$$

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which corresponds to a Kantowski-Sachs (KS) model. Eqns (2.14, 2.15) determine the constants A and B in terms of α once β and γ are specified [see CC eqn (3.18)]. However, as explained in CC, one requires β and γ to be imaginary for $\alpha > 0$, which means that V is negative (i.e. the solution is tachyonic). Furthermore, eqn (2.14) implies that M is negative for $\alpha > -1/3$, so the solution given by eqn (3.10) may be unphysical. Nevertheless, $V \to 0$ and $\xi \to 0$ providing $\alpha > -1/3$, so this solution is still formally valid. As indicated in Table (1) and discussed in CC, one can perturb this solution to obtain a 1-parameter family of solutions which asymptote towards it. This involves modifying the constants A and B, as well as adding higher order terms. If both constants c_1 and c_2 in eqn (3.8) are non-zero, the c_1 term in eqn (3.8) dominates as $z \to 0$ but the other term should not be interpreted as giving the perturbation to KS.

If $\xi \to \xi_*$ (constant), then $x \sim S^{4\alpha/(1-\alpha)}$ and eqn (2.12) becomes

$$\ddot{S} - \left(\frac{1 + 6\alpha + \alpha^2}{1 - \alpha^2}\right) \dot{S} - 2 \left(\frac{1 + 3\alpha}{1 - \alpha}\right) \frac{\dot{S}^2}{S} = 0. \tag{3.11}$$

This is again of the form (3.2) and hence has the exact solution

$$S = [c_1 + c_2 z^{(1+6\alpha+\alpha^2)/(1-\alpha^2)}]^{(\alpha-1)/(1+7\alpha)}. \tag{3.12}$$

If $c_2 = 0$, this leads to the exact static solution with (cf. Misner & Zapolsky 1964)

$$x = x_0(\alpha), \quad S = S_0(\alpha). \tag{3.13}$$

This solution is specified uniquely for a given equation of state since eqn (3.5) and eqns (2.14, 2.15) give two independent relationships between x_o and S_o. [See CC eqn (3.30) for the explicit form of eqn (3.13).] However, these values are real and the mass is positive only for $\alpha > 0$.

If $c_1 = 0$, then

$$S = Az^{-(1+6\alpha+\alpha^2)/(1+\alpha)(1+7\alpha)}, \quad x = Bz^{-4\alpha(1+6\alpha+\alpha^2)/(1-\alpha^2)(1+7\alpha)}, \tag{3.14}$$

where the constants A and B are related by eqns (2.14, 2.15) and by eqn (3.5). From eqns (2.14) and (2.16) one also has

$$V \sim z^{-(3\alpha+1)^2/(1+\alpha)(1+7\alpha)}, \quad M \sim z^{2(3\alpha+1)(1+6\alpha+\alpha^2)/(1-\alpha^2)(1+7\alpha)}, \tag{3.15}$$

so this is only consistent with the assumption that $V \to 0$ as $z \to 0$ if $-1 < \alpha < -1/7$. However, eqns (3.5) and (3.14) require $-2\alpha(1 + 3\alpha)^2/(1 - \alpha^2)(1 + 7\alpha) = (1 + \alpha)\gamma^{-2}A^{-4}B^{-(1-\alpha)/\alpha}$, and this is only satisfied for $-1 < \alpha < -1/7$ if γ (and

hence β) are imaginary. In this case, M is positive but V is negative (as in the KS case), which might be regarded as unphysical. However, one can show that eqns (2.14, 2,15) are still satisfied, and indeed give the same relationship between A and B as above, providing $-1/7 > \alpha > 2\sqrt{2} - 3 = -0.17$. Since this does not seem to correspond to any well-known solution, we describe it as asymptotically "X" in Table (1). If both constants c_1 and c_2 in eqn (3.12) are non-zero, then the c_1 term dominates as $z \to 0$ for $\alpha > 2\sqrt{2} - 3$. Although one might regard this as an asymptotically static solution, it is not equivalent to a perturbed static solution.

If $\xi \to \infty$, it can be shown that there is no consistent solution.

• $V \to \infty$ as $z \to \infty$

In this case, eqn (3.1) implies

$$\frac{\dot{x}}{x} = 2\alpha \frac{\dot{S}}{S} + \left\{ \frac{2\alpha^2}{1+\alpha} \frac{1}{V^2} \right\} - \xi, \quad \xi \equiv \alpha(1+\alpha)\beta^2 x^{(\alpha-1)/\alpha} z^{-2(1-\alpha)/(1+\alpha)}, \quad (3.16)$$

where we have used eqn (2.16) to eliminate V in the expression for ξ The term in curly brackets is included to allow for the possibility that $\dot{x}/x \to 0$ and $\dot{S}/S \to 0$. If $\xi \to 0$, this implies $x \sim S^{2\alpha}$ and so eqn (2.12) becomes

$$\ddot{S} + \left(\frac{1-\alpha}{1+\alpha} \right) \dot{S} - 2\alpha \frac{\dot{S}^2}{S} = 0. \quad (3.17)$$

Eqn (3.2) implies that the solution of this equation is given by

$$S = [c_1 + c_2 z^{(\alpha-1)/(1+\alpha)}]^{1/(1-2\alpha)}. \quad (3.18)$$

The case $c_2 = 0$ gives the the exact static model, in which case x_o and S_o have the values indicated by eqn (3.13) and one needs the term in curly brackets in eqn (3.16). The case $c_1 = 0$ gives another 1-parameter solution,

$$S = A z^{(\alpha-1)/(1+\alpha)(1-2\alpha)}, \quad x = B z^{2\alpha(\alpha-1)/(1+\alpha)(1-2\alpha)}, \quad (3.19)$$

where A and B are related by eqns (2.14, 2.15). However, this leads to

$$V \sim z^{(1-\alpha)/(1+\alpha)(1-2\alpha)}, \quad \xi \sim z^{2\alpha(1-\alpha)/(1+\alpha)(1-2\alpha)}, \quad (3.20)$$

so $V \to \infty$ and $\xi \to 0$ as $z \to \infty$ only for $\alpha < 0$. In this case, eqns (2.14, 2.15) can be satisfied only if β and γ are imaginary, so that V is negative, but M is still positive. Since this does not seem to correspond to any well-known solution, we describe it as asymptotically "Y" in Table (1).

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For $c_1 \neq 0$ and $c_2 \neq 0$, eqn (3.18) leads to a solution

$$S = A + Bz^{-(1-\alpha)/(1+\alpha)}, \quad x = C + Dz^{-(1-\alpha)/(1+\alpha)}. \qquad (3.21)$$

In this case, the term in curly brackets in eqn (3.16) can still be neglected since $\dot{x}/x$ and $\dot{S}/S$ go to zero more slowly than V^{-2}. There are two relationships between the four integration constants A, B, C, D, eqn (3.21) requiring $D = 2\alpha B$ and eqns (2.14, 2.15) giving another relationship, so these solutions are described by two independent parameters. Note that S tends to a constant as $z \to \infty$ but this constant is different from the value S_0 given by eqn (3.13), so these solutions are described as asymptotically "quasi-static". Indeed eqns (3.21) corresponds to the full family of such solutions discussed by CC and also (implicitly) by Ori & Piran (1990) and Foglizzo & Henriksen (1992).

If $\xi \to \xi_*$ (constant), then $x \sim z^{-2\alpha/(1+\alpha)}$ and so eqn (3.18) again applies. For $\alpha > -1/3$, this leads to either the Friedmann solution, given by eqn (3.9), or the KS solution, given by eqn (3.10). Although the general solution with $c_1 \neq 0$ and $c_2 \neq 0$ can be written as

$$S = Az^{-2/3(1+\alpha)}[1 + Bz^{-(1+3\alpha)/(1+\alpha)}] \qquad (3.22)$$

and might therefore be regarded as being asymptotic to the Friedmann solution, the second term is not equivalent to the first order perturbation of the Friedmann solution. CC show that this pertubation goes like $z^{-2(1+3\alpha)/3(1+\alpha)}$ and this dominates the second term in eqn (3.22) as $z \to \infty$.

If $\xi \to \infty$, there is no consistent solution.

● $V \to \infty$ as $z \to 0$

In this case, eqn (3.16) applies. If $\xi \to 0$, one again obtains eqn (3.18). However, the static solution with $c_2 = 0$ has $V \to 0$ for all α, while eqn (3.20) implies that no solution with $c_1 = 0$ has both $V \to \infty$ and $\xi \to 0$. If $\xi \to \xi_*$, one gets both a Friedmann solution of the form (3.9) and a KS solution of the form (3.10) with $V \to \infty$ and $\xi \to 0$ providing $-1 < \alpha < -1/3$. Furthermore, unlike the $\alpha > 0$ case, the KS solution is now physical in that M and V are positive. If $\xi \to \infty$, there is no consistent solution.

● $V \to 0$ as $z \to \infty$

In this case, eqn (3.5) applies. If $\xi \to 0$, one again obtains eqn (3.8) and one gets both a Friedmann solution and a (physical) KS solution with $V \to 0$ and $\xi \to 0$ providing $-1 < \alpha < -1/3$. If $\xi \to \xi_*$, one gets a solution of the form

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(3.12). However, the static solution with $c_2 = 0$ has $V \to \infty$ for all α, while eqn (3.5) cannot be satisfied for $c_1 = 0$ because

$$\frac{\dot{x}}{x} + \frac{2\alpha}{1+\alpha} = -\frac{2\alpha(1+3\alpha)^2}{(1-\alpha^2)(1+7\alpha)} \tag{3.23}$$

and therefore has the same sign as ξ. If $\xi \to \infty$, there is no consistent solution.

- $V \to \infty$ as $z \to z_*$

In this case, eqn (3.1) implies

$$\frac{\dot{x}}{x} = 2\alpha\frac{\dot{S}}{S} - \xi, \quad \xi \equiv \alpha(1+\alpha)\beta^2 x^{(\alpha-1)/\alpha} z_*^{-2(1-\alpha)/(1+\alpha)}, \tag{3.24}$$

(cf. eqn (3.16) without the term in curly brackets). If $\xi \to \infty$, then $\dot{x}/x$, $\dot{S}/S$ and ξ must all diverge at the same rate (i.e. there is no self-consistent solution with $\dot{x}/x << \dot{S}/S$ or $\dot{x}/x >> \dot{S}/S$). Hence,

$$\frac{\dot{x}}{x} \sim x^{(\alpha-1)/\alpha}, \tag{3.25}$$

which integrates to

$$x = (d_1 + d_2 lnz)^{\alpha/(1-\alpha)}. \tag{3.26}$$

To ensure $\xi \to \infty$ as $z \to z_*$, one requires $x \to 0$ for $\alpha > 0$ or $x \to \infty$ for $\alpha < 0$. In both cases, one needs $d_1 = -d_2 lnz_*$ and hence

$$x \sim [ln(z/z_*)]^{\alpha/(1-\alpha)}. \tag{3.27}$$

Defining

$$L \equiv ln(z/z_*), \tag{3.28}$$

it is easy to show that the only consistent solution is then

$$S = AL^{2/3(1-\alpha)}, \quad x = BL^{\alpha/(1-\alpha)} \tag{3.29}$$

where A and B are constants related by eqns (2.14, 2.15) and eqn (3.24). Thus the scale factor goes to zero, while the density diverges for $\alpha > 0$ and goes to zero for $\alpha < 0$. However, we also have

$$V \sim L^{-(1+3\alpha)/3(1-\alpha)} \tag{3.30}$$

and this diverges as $z \to z_*$ (as required) only for $\alpha > -1/3$. Eqn (2.14) implies

$$M \sim L^{-2/3(1+\alpha)}, \tag{3.31}$$

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so $M \to \infty$ at $z = z_*$ and $MS \to$ constant. Eqn (2.14) also implies that M is positive providing the sign of $\dot{S}/S$ is positive, so this requires that $z \to z_*$ from *above*. Note that eqn (3.24) yields the same relation between A, B and z_* as eqns (2.14, 2.15), so these solutions are described by two independent parameters. Eqns (2.10) and (2.11) imply that the metric components are

$$e^{\nu} \sim L^{\alpha/(1-\alpha)} \to 0, \quad e^{\lambda} \sim L^{-\alpha/3(1-\alpha)} \to \infty, \quad R \sim L^{2/3(1-\alpha)} \to 0. \qquad (3.32)$$

corresponding to a singularity of infinite density (cf. Schwarzschild).

If $\xi \to 0$ or ξ_* (constant), it can be shown that there is no consistent solution.

• $V \to V_*$ as $z \to z_*$
In this case, eqn (3.1) implies

$$\frac{\dot{x}}{x} = \frac{2V^2\alpha}{V^2 - \alpha}\frac{\dot{S}}{S} + \frac{2\alpha^2}{1+\alpha}\frac{1}{V_*^2 - \alpha} - \xi, \quad \xi \equiv \frac{\alpha(1+\alpha)\beta^2V_*^2}{V_*^2 - \alpha}z_*^{-2(1-\alpha)/(1+\alpha)}x^{(\alpha-1)/\alpha}, \qquad (3.33)$$

where V rather than V_* appears in the first term on the right because the product $(V^2 - V_*^2)(\dot{S}/S)$ may tend to a finite limit. If $\xi \to 0$, this implies

$$x \sim S^{2V_*^2\alpha/(V_*^2-\alpha)} \qquad (3.34)$$

providing $\dot{x}/x$ and $\dot{S}/S$ diverge. However, unless $\dot{V}/V \to \infty$, eqn (2.16) also implies $x \sim S^{2\alpha/(1-\alpha)}$, so we require $V_*^2 = 1$. Eqn (2.12) then becomes

$$\ddot{S} + \left(\frac{1 - 4\alpha - \alpha^2}{1 - \alpha^2}\right)\dot{S} - \frac{4\alpha}{1-\alpha}\frac{\dot{S}^2}{S} = 0. \qquad (3.35)$$

This is of the form of eqn (3.2), so the exact solution for $\alpha \neq 1/5$ and $\alpha \neq \sqrt{5}-2$ is given by

$$S = [c_1 + d(z/z_*)^{(\alpha^2+4\alpha-1)/(1-\alpha^2)}]^{(1-\alpha)/(1-5\alpha)}. \qquad (3.36)$$

In order to have $\xi \to 0$ as $z \to z_*$, eqn (3.33) implies that one needs S to go to infinity, whatever the sign of α. One therefore requires $d = -c_1$ and $\alpha > 1/5$. This excludes $\alpha < 0$ and also the special cases $\alpha = 1/5$ and $\alpha = \sqrt{5} - 2$. Noting that to first order

$$1 - \left(\frac{z}{z_*}\right)^{\frac{\alpha^2+4\alpha-1}{1-\alpha^2}} \sim \left(\frac{\alpha^2 + 4\alpha - 1}{1 - \alpha^2}\right)L, \qquad (3.37)$$

we obtain the asymptotic form

$$S = AL^{(1-\alpha)/(1-5\alpha)}, \quad x = BL^{2\alpha/(1-5\alpha)} \qquad (3.38)$$

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where A and B are constants. Thus the scale factor diverges and the density goes to zero. The condition $V_* = 1$ gives a relationship between the constants A, B and z_*, so these solutions are described by two parameters.

Eqn (2.14) implies that

$$M \sim L^{(1-\alpha)/(5\alpha-1)}, \tag{3.39}$$

so these solutions have zero mass at $z = z_*$ and MS tends to a constant. Furthermore the sign of $\dot{S}/S$ implies that z must approach z_* from *below* in order that the mass be positive. Eqn (2.15) can be written as

$$M = \frac{1}{2} + \frac{1}{2}\gamma^2 x^{-2/\alpha} S^6 \left\{ \left(\frac{\dot{S}}{S}\right)^2 (V^2 - 1) - \frac{2\dot{S}}{S} - 1 \right\}. \tag{3.40}$$

Since $x^{-2/\alpha} S^6 \sim L^{2(3\alpha-1)/(5\alpha-1)}$ goes to infinity for $\alpha < 1/3$ and zero for $\alpha > 1/3$, one requires the term in curly brackets to go to to zero and infinity, respectively, in these two cases. However, the last term in eqn (3.40) also scales as

$$L^{(\alpha-1)/(5\alpha-1)} \left[\frac{\dot{S}}{S}(V^2 - 1) - 2 - \frac{S}{\dot{S}} \right], \tag{3.41}$$

so we need the term in square brackets to go to zero as $L^{(1-\alpha)/(5\alpha-1)}$ in both cases. We therefore need

$$\frac{\dot{S}}{S}(V^2 - 1) \to 2. \tag{3.42}$$

Using eqns (2.16) and (3.33), together with the relation

$$\frac{V^2}{V^2 - \alpha} \approx \frac{1}{1-\alpha} \left[1 - \frac{\alpha}{1-\alpha}(V^2 - 1) \right], \tag{3.43}$$

one then obtains

$$\frac{\dot{V}}{V} \to \frac{1 - 5\alpha}{1 - \alpha}. \tag{3.44}$$

Condition (3.41) also determines the second order terms in the expressions for x and S but does not impose any further relationship between A, B and z_*. Note that the metric can be written as

$$ds^2 \sim L^{4\alpha/(1-5\alpha)}[dt^2 - dr^2 - r^2 L^{2(3\alpha-1)/(5\alpha-1)} d\Omega^2]. \tag{3.45}$$

This resembles the open Friedmann model and is related to Minkowski spacetime by a time-transformation (as in the Milne model). These solutions can

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therefore be regarded as asymptotically flat, or, more precisely, as aymptotically Schwarzschild since the mass ($\sim MS$) tends to a non-zero constant.

If $\xi \to \infty$ or constant, it can be shown that there is no consistent solution.

- $V \to V_*$ as $z \to \infty$

In this case, we require $\dot{V} \to 0$ and eqn (2.16) then implies

$$\frac{\dot{S}}{S} = \frac{1}{2}\left(\frac{1-\alpha}{\alpha}\right)\frac{\dot{x}}{x} + \frac{1}{2}\left(\frac{1-\alpha}{1+\alpha}\right). \tag{3.46}$$

The exact solution of this differential equation is

$$S^2 = S_0 z^{\frac{1-\alpha}{1+\alpha}} x^{\frac{1-\alpha}{\alpha}} \tag{3.47}$$

which can then be combined with eqn (3.1) to give

$$\frac{\dot{x}}{x} = \frac{V_*^2(1-\alpha) + 2\alpha}{(V_*^2 - 1)(1+\alpha)} - \xi', \quad \xi' \equiv \frac{(1+\alpha)\beta^2 V_*^2}{V_*^2 - 1} z^{-2(1-\alpha)/(1+\alpha)} x^{(\alpha-1)/\alpha} \tag{3.48}$$

where ξ' differs slightly from ξ. If $\xi' \to 0$, this implies

$$x = Bz^m, \quad m = \frac{V_*^2(1-\alpha) + 2\alpha}{(V_*^2 - 1)(1+\alpha)} \tag{3.49}$$

and eqn (2.16) then gives

$$S = Az^n, \quad n \equiv \frac{(1-\alpha)(V_*^2 + \alpha)}{2\alpha(1+\alpha)(V_*^2 - 1)} \tag{3.50}$$

where A and B are integration constants. For $\alpha > 0$, these solutions are consistent with the condition $\xi' \to 0$ as $z \to \infty$ providing $x \to \infty$ and, from eqn (3.48), this requires $V_*^2 > 1$. This in turn implies $S \to \infty$. Eqn (2.14) gives

$$M \sim \frac{(V^2 - \alpha)(1+\alpha)}{2\alpha(V^2 - 1)} z^{-[V_*^2(1-\alpha)+1+3\alpha]/(V_*^2-1)(1+\alpha)} \tag{3.51}$$

and so the mass is positive and tends to zero for $V_*^2 > 1$. On the other hand, eqns (2.15), (3.48) and (3.49) imply

$$M - \frac{1}{2} \sim z^{[V_*^2(1-\alpha)-\alpha(1+3\alpha)]/(V_*^2-1)\alpha(1+\alpha)} \left\{ \left(\frac{\dot{S}}{S}\right)^2 (V^2 - 1) - \frac{2\dot{S}}{S} - 1 \right\}. \tag{3.52}$$

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If the exponent of z in this expression is positive, M can go to zero only if the term in curly brackets does and this requires $\dot{S}/S \rightarrow 1/(V_* - 1)$. Equating this with the value of $\dot{S}/S$ implied by eqn (3.48) gives a quadratic equation for V_*:

$$(1 - \alpha)V_*^2 - 2\alpha(1 + \alpha)V_* - \alpha(1 + 3\alpha) = 0 \qquad (3.53)$$

with the solution

$$V_* = \frac{\alpha(1 + \alpha) + \sqrt{\alpha(\alpha^3 - \alpha^2 + 3\alpha + 1)}}{1 - \alpha}. \qquad (3.54)$$

As α decreases from 1 to 0, V_* decreases from ∞ to 0 and it exceeds 1 (as required) only for $\alpha > 1/5$. Note that eqn (3.52) implies that the exponent of z in eqn (3.51) is indeed positive (as assumed).

The value of V_* is also well-defined for $-1 > \alpha > -1/3$. However, if one seeks solutions of this kind, then the condition $\xi' \rightarrow 0$ as $z \rightarrow \infty$ implies $x \rightarrow 0$ and eqn (3.48) would then give $1 > V_*^2 > -2\alpha/(1 - \alpha)$. The lower limit is incompatible with eqn (3.53), so there are no $\alpha < 0$ solutions. One can also formally obtain a negative value of V_* by taking the negative square root in eqn (3.53). This is relevant if one is considering asymptotically KS solutions with $\alpha > 0$.

Eqns (3.48), (3.49) and (3.53) impose a relationship between the constants A and B and so these solutions are described by one-parameter and they all tend to the same value of V for a given α. The requirement that the term in curly brackets in eqn (3.52) goes to zero also determines the second order terms in the expansions for x and S. These solutions, like the previous ones, are asymptotically flat: the metric components can be expressed as

$$e^\nu \sim z^{V_*^2/(V_*^2 - 1)}, \quad e^\lambda \sim z^{1/(V_*^2 - 1)}, \quad R \sim z^{1/(V_* - 1)} \qquad (3.55)$$

and the metric can be reduced to the Minkowski form with a suitable change of coordinates.

If $\xi \rightarrow \infty$ or constant, there is no consistent solution.

• $V \rightarrow V_*$ as $z \rightarrow 0$

In this case, eqns (3.46) - (3.53) still apply but, for $\alpha > 0$, the condition $\xi' \rightarrow 0$ requires $V_*^2 < 1$. Eqn (3.50) then implies that the mass is negative unless $V_* < 1/\sqrt{\alpha}$. However, eqn (3.53) requires that one always has $V_* > 1/\sqrt{\alpha}$, thus excluding this case. For $\alpha < 0$, the condition $\xi' \rightarrow 0$ requires either $V_*^2 > 1$ or $V_*^2 < -2\alpha/(1 - \alpha)$. The first possibility is excluded because the mass is negative but, the second possibility is allowed providing $-1 < \alpha < -1/3$, so that V_* is defined.

4 Discussion

In this paper we have focussed on establishing the asymptotic forms of spherically symmetric perfect fluid similarity solutions. The results are summarized in Table 1. Only those cases that give rise to self-consistent and physically reasonable solutions have been described explicitly (none of the remaining cases give rise to self-consistent solutions). Although some of the details of the calculations have been omitted, the analysis presented can be regarded as a rigorous justification for the underlying assumption of our complete classification of positive α solutions in Carr & Coley (1999b). Although our description of these solutions has been rather brief, their physical properties have been further studied by Goliath et al. (1998a, 1998b) and Carr et al. (1999). Our analysis has covered all equations of state with $|\alpha| < 1$ but the stiff fluid case ($\alpha = 1$) has not been explicitly included (although many of the results carry through in this case); this is because different mathematical techniques are more appropriate in this case and the analysis will be presented elsewhere.

It should be stressed that our analysis is not completely rigorous since we have implicitly assumed that certain regularity conditions are satisfied. For example, in Section 3 we always presupposed that V has a well-defined asymptotic limit. Also our analysis has assumed certain properties for the derivatives of small quantities. However, an asymptotic relationship of the form $X \gg Y$ does not necessarily imply $\dot{X} \gg \dot{Y}$ asymptotically. In order to "prove" the results of Section 3 we must show that this type of behaviour is not possible and this must be done on a case-by-case basis. For example, suppose an asymptotic solution of the form $x \sim z^l$ is deduced. In principle

$$x \sim z^l(1+\varepsilon) \tag{4.1}$$

where

$$\frac{\dot{x}}{x} = l + E, \quad E \equiv \frac{\dot{\varepsilon}}{1+\varepsilon}. \tag{4.2}$$

However, the condition $\varepsilon \ll 1$ does not necessarily imply $E \ll 1$. To prove the result, we must keep ε and $\dot{\varepsilon}$ in all of the equations and show that the only self-consistent solutions must have $\varepsilon \ll 1$. Suffice it to say that all cases have been checked and the only self-consistent solutions are those given in this paper. In fact, studying solutions of the form given by eqn (4.2) is precisely what we do in CC in deriving solutions which are asymptotic to the Friedmann and Kantowski-Sachs models (although in practice this calculation was divided up into different separate cases than in that of section 3–and this served as a further check of the validity of our results).

The results of Section 3, and the existence of well-defined limits and sufficient regularity, also follow from the mathematically more rigorous dynamical systems analyses of Bogoyavlensky (1985) and Goliath et al. (1998a, 1998b), at least in the positive α case. In these analyses, monotone functions and Dulac functions were found to exist, thereby prohibiting periodic orbits and limit cycles in the corresponding phase spaces. This rules out the possibility of kinds of asymptotic behaviour different from those discussed in this paper. Also, many of the results apply to the case $\alpha = 0$ and are consistent with the results of the analytic study of the dust models (Carr 1999); indeed, this serves as a further check of the validity of our results here.

Finally, we should emphasis that our analysis has demonstrated that all self-similar spherically symmetric perfect fluid solutions are asymptotically of power-law form for $z \to \infty$ or $z \to 0$ and of log-power-law form for $z \to z_*$. This provides the basis of the work presented in our classification paper.

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